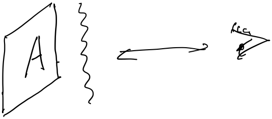
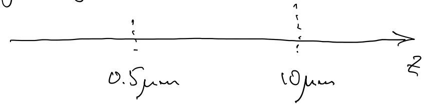
Rayleigh and Mie scattering of light

Optical scattering



(macro)

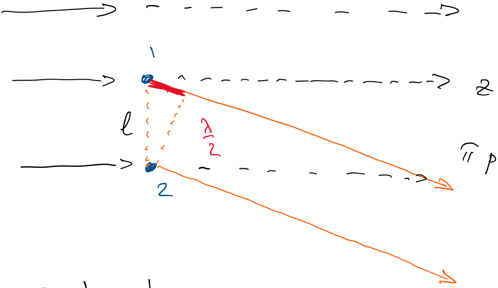
Rayleigh-Tyndall → Mie → Optical



Re-emission of secondary sources:
Huygens principle

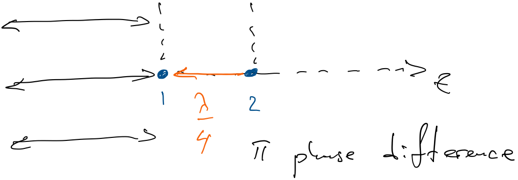
1) Ideal isotropic medium

Forward:



π phase shift

Backward:

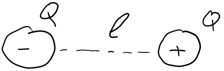


The only option is for light to go forward. Hence scattering requires any level of anisotropy.

Simplest model again will be re-emission by the dipole.

First and foremost we need to remember that e/m wave can be emitted only by accelerating charge. Steady staying or moving charge with $V = \text{const}$ is not emitting e/m wave.

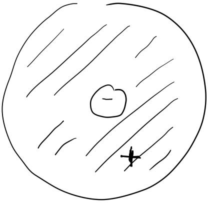
Obviously, we need to take the simplest model – dipole oscillator:



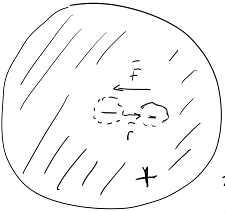
P – dipole model of such oscillator

$$\boxed{P = Q\ell}$$

Now, how can we envision atom as dipole oscillator? The simplest „artificial" solution has been proposed by Thompson and is called Thompson atom model.



According to Thompson, atom is positive cloud with negative electron in a center



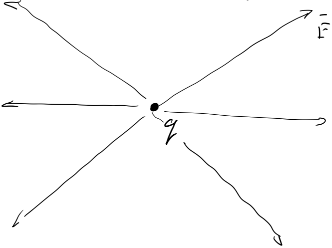
If we apply some force to displace electron from the center, then another force appear that returns the electron back to its equilibrium position.

$$\bar F = -k\bar r$$

k is restoring coefficient.

Let's see how we can visualize emission of e/m wave using Thompson model:

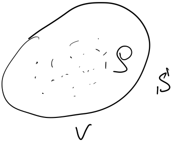
$t=0$, the charge is still



Its electric field will be represented by field lines

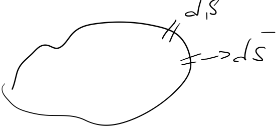
Let's see how $\bar{E}$ changes with $\bar{r}$

$$\mathrm{div}\,\bar{E} = \frac{\rho}{\varepsilon_0}$$



$$\int_V \mathrm{div}\,\bar{E}\,dV = \frac{1}{\varepsilon_0}\int_V \rho\,dV$$ Ostrogradsky Gauss

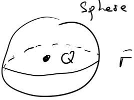
$$\oint_S \bar{E}\,d\bar{S} = \frac{1}{\varepsilon_0}\int \rho\,dV$$



The flux of electric field through closed surface S is determined by the total charge within V.

$$\oint_S \bar{E}\,d\bar{S} = \frac{1}{\varepsilon_0}\int \rho\,dV = \frac{Q}{\varepsilon_0}$$

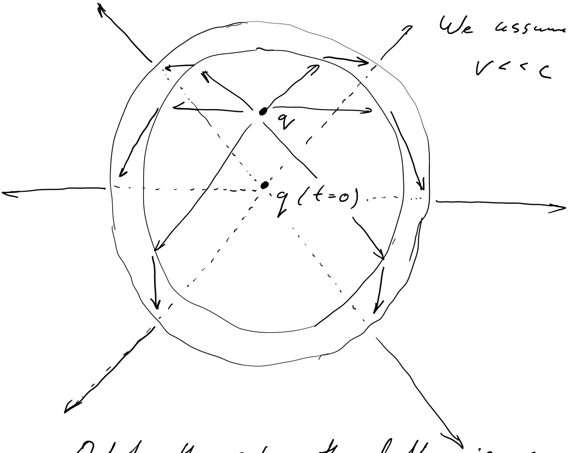
$$4\pi r^2(\bar{E}\bar{r}) = \frac{Q}{\varepsilon_0}$$



$$\bar{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\bar{r}$$

At certain moment the charge is moved by force $\bar{F}$ for the time τ and then left going with certain speed $V = \text{const}$ as force is not applied anymore.

We assume that $v << c$

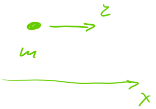


Outside this sphere the field is similar as at $t=0$. Inside the sphere it is also static as $v = \text{const}$.

The breaks of force lines that travel away from the charge is actually the wave. Now we know, that if we apply restoring force similar to $\bar{F} = -k\bar{z}$, then the oscillator will have harmonic oscillations.

Little mathematical walkaround

$\bar{F} = -k\bar{z}$



Then from the second Newton's law:

$$-kx = ma$$

$$m\frac{\partial^2 x}{\partial t^2} + kx = 0 \Rightarrow \frac{\partial^2 x}{\partial t^2} + w^2 x = 0$$

Proof that

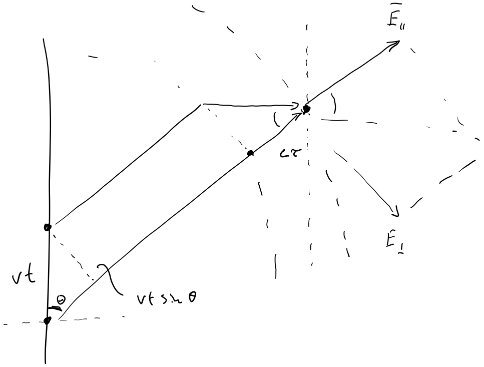
$x(t) = A\cos(\omega t + \varphi)$

Let's have a look at one of the lines that have angle θ with respect to the charge movement.

τ - time of charge brief acceleration

t - time charge moves with $v = const$ after

$t >> \tau$



As it is clear

$$\frac{E_\perp}{E_\parallel} = \frac{vt\sin\theta}{c\tau}$$ (Explain!)

Since at distances $r = ct$

$$E_\parallel = \frac{q}{4\pi\varepsilon_0 r^2}$$ and $r = ct$

$$E_\perp = \frac{qv}{4\pi\varepsilon_0 c^3 t\tau} \sin\theta$$

Since $\frac{v}{\tau} = a$, acceleration of moving charge, when the force was applied:

$$E_\perp = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qa}{c^2 r} \sin\theta$$

What does it mean?

1) $E_\perp$ is dependent on a. However a is the acceleration that happened at $t=0$. The wave emitted at $t=0$ will reach the observation point at $\frac{r}{c}$.

$$a(t - r/c)$$

We can harmonically oscillate the charge:

$$z = z_0 \cos\omega t$$

$$a(t) = \frac{\partial^2 z}{\partial t^2} = -\omega^2 z_0 \cos\omega t$$

$$E_\perp = \frac{1}{4\pi\varepsilon_0 c^2 r} Q\omega^2 z_0 \cos\left(\omega\left(t - \frac{r}{c}\right)\right)\sin\theta = A\cos(\omega t - kz)$$

2) $E_\perp \sim \frac{1}{r}$

$$E_\perp = \frac{1}{4\pi\varepsilon_0} \frac{qa}{c^2 r} \sin\theta = \underbrace{\frac{1}{4\pi\varepsilon_0} \frac{Q z_0 \omega^2}{c^2 r}}_{A \sim \frac{1}{r}} \cos(\omega t - kr)$$

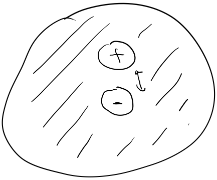
Moreover

$$E_{\parallel} \sim \frac{1}{r^2} \to 0$$

$$\bar{E}_{\perp} \sim \frac{1}{r}$$

It means it is splerical wave

We can add another charge $-q$ to the center of coordinates. However, since it is static, it will only influence



$\bar{E}_{\parallel}$, that is decaying as $\frac{1}{r^2}$, while $\bar{E}_{\perp} \sim \frac{1}{r}$

From this point it is not important if we move charge q or $-q$, or, actually change the charge itself as $q = q_0 \cos \omega t$ while keeping its position fixed.

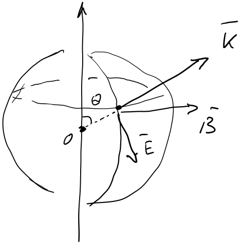
Changing q -> radio antenna

Changing z_0 -> atom emission

3) $E_\perp \sim \sin\Theta$

It means the oscillator emits more in direction perpendicular to the movement.

When $r >> \lambda$ spherical wave can be seen as plane:



In this polar coordinates

$$\begin{cases} E_r = 0 \\ E_\theta = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\ddot{P}}{c^2 r} \sin\Theta \\ E_\varphi = 0 \end{cases}$$

$\underline{P} = Q\ell$

$$\begin{cases} B_r = 0 \\ B_\theta = 0 \\ B_\varphi = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\ddot{P}}{c^2 r} \sin\Theta \end{cases} \Rightarrow \bar{B} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{c^2 r} \left[\ddot{\bar{P}}\, \bar{r}_0 \right]$$

$\bar{r}_0 = \frac{\bar{r}}{r}$

Thus $\boxed{\bar{E} = c\left[\bar{B}\, \bar{r}_0\right]}$

It is important to remember, that although field of emission of dipole oscillator is sphere, it does not possess spherical symmetry. Modulus of $\bar{E}$ and $\bar{B}$ are different at different positions of sphere and depend on θ.

The energy flux will be:

$$I = \langle S \rangle = \varepsilon_0 c \langle [\bar{E}\bar{B}] \rangle =$$

$$= \varepsilon_0 c^2 \langle EB \rangle = \varepsilon_0 c \langle E^2 \rangle = \varepsilon_0 c \frac{\langle \ddot{p}^2 \rangle \sin^2\theta}{(4\pi)^2 \varepsilon_0^2 c^4 r^2}$$

Since $p = p_0 \cos(\omega t)$

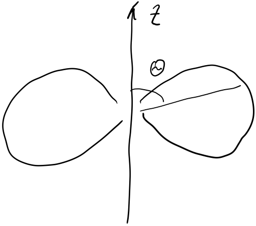
$$p(t') = p_0 \cos\left(\omega\left(t - \frac{r}{c}\right)\right) = p_0 \cos(\omega t - kr)$$

$$\ddot{p} = \omega^2 p_0 \cos(\omega t - kr)$$

$$\Downarrow$$

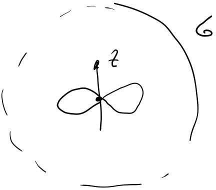
$$\langle \ddot{p}^2 \rangle = \frac{1}{2} \omega^4 p_0^2$$

$$\boxed{I = \frac{\omega^4 p_0^2}{32\pi^2 \varepsilon_0 c^3 r^2} \sin^2\theta}$$



Now, let's determine the whole energy of emission

$$P = \int_\sigma I \, d\sigma =$$



$$= \int_0^{2\pi}\int_0^{\pi} I \, r^2 \sin\theta \, d\theta \, d\varphi =$$

integrating on θ

integrating on φ

$$= \frac{\omega^4 p_0^2}{32\pi^2 \varepsilon_0 c^3} \, 2\pi \underbrace{\int_0^{\pi} \sin^3\theta \, d\theta}_{\frac{4}{3}} = \frac{\omega^4 p_0^2}{12\pi \varepsilon_0 c^3}$$

What does it mean?

① $P \sim p_0^2$

$P \sim \omega^4 \sim \frac{1}{\lambda^4}$

That is why sky is blue during day and red on the sunset.

② P does not depend on r. This means that energy flux through any sphere R is the same. If there is no conductors or charges in the space, there is no way to lose energy or transfer it to another form.

Demonstration

What if particle is bigger than single dipole, but still not macro?

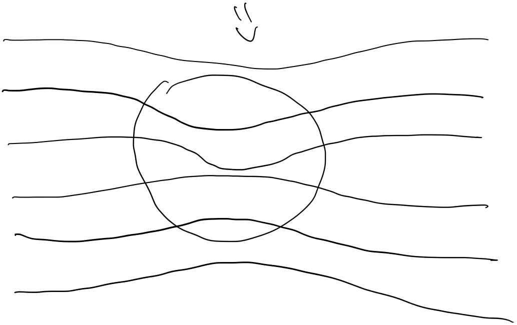
Mie theory (very difficult)

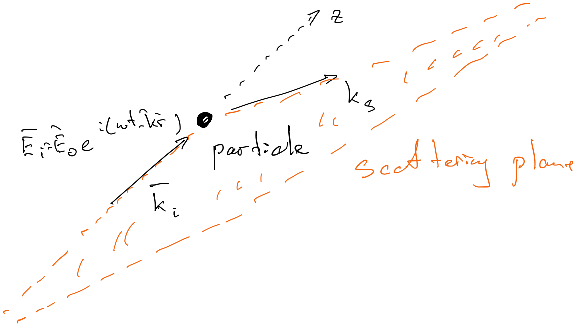
1908 (Gustav Mie)

$$\begin{cases} \operatorname{div} \bar{E} = \dfrac{\rho}{\varepsilon_0} \\ \operatorname{div} \bar{B} = 0 \\ \operatorname{rot} \bar{E} = -\dfrac{\partial \bar{B}}{\partial t} \\ \operatorname{rot} \bar{B} = \bar{j} + \dfrac{\partial \bar{E}}{\partial t} \end{cases}$$

+ boundary conditions

full discontinuity of $\bar{E}$ and $\bar{B}$





Amplitud scattering matrix

$$\begin{pmatrix} E_{s\parallel} \\ E_{s\perp} \end{pmatrix} = \frac{e^{i(\omega t - \bar{k}\bar{r})}}{i\bar{k}\bar{r}} \begin{pmatrix} S_{\parallel} & S_{s1} \\ S_{s2} & S_{\perp} \end{pmatrix} \begin{pmatrix} E_{i\parallel} \\ E_{i\perp} \end{pmatrix}$$

Spherical wave

$S_{s1} = S_{s2} = 0$ (sphere)   Jones matrix

$S \to f(d, n, \lambda)$

$$E_{\theta_{\parallel}} = e^{i\omega t}\cos\theta \sum_{1}^{\infty} -\alpha_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\, h_n^{(2)}(\hat{k}\hat{r})$$

$$E_{\theta_{\perp}} = e^{i\omega t}\sin\theta \sum_{1}^{\infty} -\beta_n(-i)^n \frac{2n+1}{n(n+1)} P_n^1(\cos\theta)\, h_n^{(2)}(\hat{k}\hat{r})$$

$P_n^1(\cos\theta)$ – Legendre polynomial

$h_n^{(2)}(\hat{k}\hat{r})$ – Bessel function

$$h_n^{(2)}(\hat{k}\hat{r}) = \frac{i^{n+1}}{\hat{k}\hat{r}} e^{-i\hat{k}\hat{r}}$$

Demonstration

The magic of Mie theory

https://omlc.org/calc/mie_calc.html